International Journal of Engineering, Science and Mathematics
Vol. 7 Issue 5, May 2018,
ISSN: 2320-0294 Impact Factor: 6.765
Journal Homepage: http://www.ijesm.co.in, Email: ijesmj@gmail.com
Double-Blind Peer Reviewed Refereed Open Access International Journal - Included in the International Serial Directories Indexed & Listed at: Ulrich's Periodicals Directory ©, U.S.A., Open J-Gage as well as in Cabell's Directories of Publishing Opportunities, U.S.A

On the central index of composite entire functions

Sanjib Kumar Datta*1
BananiDutta2**

Abstract

In this paper we investigate the composition of two entire functions with their corresponding left and right factors in the light of their central indices from the view point of the joint effect of slowly changing functions of higher index and a special type of non-deceasing unbounded function ψ.

AMS Subject Classification(2000) : 30D35 , 30D30.

Keywords:

Entire function ;
Composition;
Growth;
Order (lower order);
Maximum term;
Central index;
Slowly changing function;
Non-deceasing unbounded function ψ.

Author correspondence:

[1&2]Department of Mathematics,University of Kalyani
P.O.-Kalyani, Dist.- Nadia,PIN-741235.

1. Introduction

Let f be an entire function defined in the open complex plane C. The maximum term $\mu(r,f)$ of $f=\sum_{n=0}^{\infty} a_n z^n$ on |z|=r is defined by $\mu(r,f)=\max_{n\geq 0}\left(|a_n| r^n\right)$. To start this paper we just recall the following:

Let f be an entire function defined in the open complex plane C .The central index $\nu_f(r)=\nu(r,f)$ is the greatest exponent m such that $|a_m| r^m=\nu(r,f)$. We note that $\nu(r,f)$ is a real and non-decreasing function of r.

The following definitions are well known.

Definition 1 The order ρ_f and lower order λ_f of an entire function f is defined as

$$\rho_f=\limsup_{r\to\infty}\frac{\log^{[2]} M(r,f)}{\log r} \text{ and } \lambda_f=\liminf_{r\to\infty}\frac{\log^{[2]} M(r,f)}{\log r}$$

where $\log^{[k]} x = \log\left(\log^{[k-1]} x\right), for k = 1,2,...., and \log^{[0]} x = x.$

Definition 2 The hyper order $\overline{\rho}_f$ and hyper lower order $\overline{\lambda}_f$ of an entire function f is defined as follows

$$\overline{\rho}_f = \limsup_{r\to\infty} \frac{\log^{[3]} M(r,f)}{\log r} \text{ and } \overline{\lambda}_f = \liminf_{r\to\infty} \frac{\log^{[3]} M(r,f)}{\log r}.$$

Since for $0 \le r < R,$

$$\nu(r,f) \le M(r,f) \le \frac{R}{R-r}\nu(R,f).$$

,

it is easy to see that

$$\rho_f = \limsup_{r\to\infty} \frac{\log^{[2]} \nu(r,f)}{\log r} \text{ and } \lambda_f = \liminf_{r\to\infty} \frac{\log^{[2]} \nu(r,f)}{\log r}.$$

Now let us define another function :

Let $\Psi : [0,\infty) \to (0,\infty)$ be a non-decreasing unbounded function, satisfying the following two conditions:

$$(i) \lim_{r\to\infty} \frac{\log^{[p]} r}{\log^{[q]} \Psi(r)} = 0$$

$$(ii) \lim_{r\to\infty} \frac{\log^{[q]} \Psi(\alpha r)}{\log^{[q]} \Psi(r)} = 1$$

for some α >1.

Now we will define the classical definitions of growth indicators of f with respect to central index $\nu(r,f)$, with the help of the function Ψ.

Using the concept of central index we may reframe the following definitions as follows :

Definition 3 The Ψ - order $\rho_{f,\Psi}$ and lower Ψ -order $\lambda_{f,\Psi}$ of an entire function f is defined as follows:

$$\rho_{f,\Psi} = \limsup_{r\to\infty} \frac{\log^{[2]} \nu(r,f)}{\log \Psi(r)} \text{ and } \lambda_{f,\Psi} = \liminf_{r\to\infty} \frac{\log^{[2]} \nu(r,f)}{\log \Psi(r)}$$

where $\log^{[k]} x = \log\left(\log^{[k-1]} x\right), for k = 1,2,...., and \log^{[0]} x = x.$

Definition 4: the hyper Ψ -order $\overline{\rho}_{f,\Psi}$, the hyper Ψ -order $\overline{\lambda}_{f,\Psi}$ of f is defined by

$$\overline{\rho}_{f,\Psi} = \limsup_{r\to\infty} \frac{\log^{[3]} \nu(r,f)}{\log \Psi(r)} \text{ and } \lambda_{f,\Psi} = \liminf_{r\to\infty} \frac{\log^{[3]} \nu(r,f)}{\log \Psi(r)}.$$

Somasundaram and Thamizharasi ([4]) introduced the notion of L-order ,L-lower order and L-type for entire functions where L=L(r) is a positive continuous function increasing slowly i.e. L(ar) ~ L(r) as r tending to infinity for every constant 'a'.

Now we will define the L-order(L-lower order)with respect to the function Ψ.

.

Definition 5 {cf. [4]} The L-Ψ- order $\rho^{L}_{f,\Psi}$ and L- lower Ψ -order $\lambda^{L}_{f,\Psi}$ of an entire function f is defined as follows:

$$\rho^{L}_{f,\Psi}=\limsup_{r\to\infty}\frac{\log^{[2]}\nu(r,f)}{\log[\Psi(r)L(r)]} \text{ and } \lambda^{L}_{f,\Psi}=\liminf_{r\to\infty}\frac{\log^{[2]}\nu(r,f)}{\log[\Psi(r)L(r)]}.$$

When f is meromorphic, then

$$\rho^{L}_{f,\Psi}=\limsup_{r\to\infty}\frac{\log T(r,f)}{\log[\Psi(r)L(r)]} \text{ and } \lambda^{L}_{f,\Psi}=\liminf_{r\to\infty}\frac{\log T(r,f)}{\log[\Psi(r)L(r)]}.$$

Definition 6 {cf. [4]} The L-Ψ- type $\sigma^{L}_{f,\Psi}$ of an entire function f is

$$\sigma^{L}_{f,\Psi}=\limsup_{r\to\infty}\frac{\log \nu(r,f)}{[\Psi(r)L(r)]^{\rho^{L}_{f,\Psi}}}, \quad 0<\rho^{L}_{f,\psi}<\infty.$$

When f is meromorphic, then

$$\sigma^{L}_{f,\Psi}=\limsup_{r\to\infty}\frac{T(r,f)}{[\Psi(r)L(r)]^{\rho^{L}_{f,\Psi}}}, \quad 0<\rho^{L}_{f,\psi}<\infty.$$

Definition 7 : The (p,q)$^{th}-\psi$ order and lower (p,q)$^{th}-\psi$ order of an entire function f respectively as follows:

and $$\rho_{f,\Psi}(p,q)=\limsup_{r\to\infty}\frac{\log^{[p]}\nu(r,f)}{\log^{[q]}\Psi(r)}$$

$$\lambda_{f,\Psi}(p,q)=\liminf_{r\to\infty}\frac{\log^{[p]}\nu(r,f)}{\log^{[q]}\Psi(r)}.$$

wherep,q are integers with p>q.

when f is meromorphic one can easily verify that

$$\rho_{f,\Psi}(p,q)=\limsup_{r\to\infty}\frac{\log^{[p-1]}\nu(r,f)}{\log^{[q]}\Psi(r)}$$

$$\lambda_{f,\Psi}(p,q)=\liminf_{r\to\infty}\frac{\log^{[p-1]}\nu(r,f)}{\log^{[q]}\Psi(r)}.$$

where p, q are positive integers with p>q.

With the notion of slowly changing function one can easily define the following:

Definition 8 : The L-(p,q)$^{th}-\psi$ order and L- lower (p,q)$^{th}-\psi$ order of an entire function f respectively as follows:

$$\rho^{L}_{f,\Psi}(p,q)=\limsup_{r\to\infty}\frac{\log^{[p]}\nu(r,f)}{\log^{[q]}[\Psi(r)L(r)]}$$

$$\lambda^{L}_{f,\Psi}(p,q)=\liminf_{r\to\infty}\frac{\log^{[p]}\nu(r,f)}{\log^{[q]}[\Psi(r)L(r)]}.$$

The more generalized concept of L-order and L-type of entire and meromorphic functions are L*-order and L*-type with respect to Ψ, respectively .Their definitions are as follows :

Definition 9 : The L$^{*}-\psi$ order , L*- lower $-\psi$ order and L$^{*}-\psi$ type of an meromorphic function f are defined by:

$$\rho^{L^*}_{f,\Psi} = \limsup_{r\to\infty} \frac{\log T(r,f)}{\log[\Psi(r)e^{L(r)}]}$$
$$\lambda^{L^*}_{f,\Psi} = \liminf_{r\to\infty} \frac{\log T(r,f)}{\log[\Psi(r)e^{L(r)}]}.$$

and
$$\sigma^{L^*}_{f,\Psi} = \limsup_{r\to\infty} \frac{T(r,f)}{\left[\Psi(r)e^{L(r)}\right]^{\rho^{L^*}_{f,\Psi}}} \quad , 0<\rho^{L^*}_{f,\Psi}<\infty.$$

When f is entire, one can easily verify that

$$\rho^{L^*}_{f,\Psi} = \limsup_{r\to\infty} \frac{\log^{[2]} \nu(r,f)}{\log[\Psi(r)e^{L(r)}]}$$
$$\lambda^{L^*}_{f,\Psi} = \liminf_{r\to\infty} \frac{\log \nu(r,f)}{\log[\Psi(r)e^{L(r)}]}.$$

And
$$\sigma^{L^*}_{f,\Psi} = \limsup_{r\to\infty} \frac{\log \nu(r,f)}{\left[\Psi(r)e^{L(r)}\right]^{\rho^{L^*}_{f,\Psi}}} \quad , 0<\rho^{L^*}_{f,\Psi}<\infty.$$

In view of the notion of central index of entire functions we may state the following definition.

Definition 10 : The L^*-(p,q)$^{th}-\psi$ order and L^*- lower (p,q)$^{th}-\psi$ order of an entire function f are respectively as follows:

$$\rho^{L^*}_{f,\Psi}(p,q) = \limsup_{r\to\infty} \frac{\log^{[p]} \nu(r,f)}{\log^{[q]}[\Psi(r)e^{L(r)}]}$$
$$\lambda^{L^*}_{f,\Psi}(p,q) = \liminf_{r\to\infty} \frac{\log^{[p]} \nu(r,f)}{\log^{[q]}[\Psi(r)e^{L(r)}]}.$$

When f is meromorphic , then $\rho^{L^*}_{f,\Psi}(p,q)$ and $\lambda^{L^*}_{f,\Psi}(p,q)$ cannot be defined in the above way.

In the paper we further investigate the comparative growths of two entire functions with their corresponding left and right factors with respect to central index on the basis of L^- (p,q)$^{th}-\psi$ -order(lower order) and L^* (p,q)$^{th}-\psi$ - order(lower order), where p, q are positive integers and p>q.

2. Results and Analysis.

Theorem 1 : Let f and g be two entire functions such that $0<\lambda^{L}_{fog,\Psi}(p,q)\le\rho^{L}_{fog,\Psi}(p,q)<\infty$ and $0<\rho^{L}_{g,\Psi}(m,q)<\infty$, where p , q, m are positive integers such that q < min{p, m}. Then for any positive integer A,

$$(i) \liminf_{r\to\infty} \frac{\log^{[p]} \nu(r,fog)}{\log^{[m]} \nu(r^A,g)} \le \frac{\rho^{L}_{fog,\Psi}(p,q)}{\rho^{L}_{g,\Psi}(m,q)} \le \limsup_{r\to\infty} \frac{\log^{[p]} \nu(r,fog)}{\log^{[m]} \nu(r^A,g)}.$$

Further if $\lambda^{L}_{g,\Psi}(m,q)>0$ then

$$(ii) \frac{\lambda^{L}_{fog,\Psi}(p,q)}{\rho^{L}_{g,\Psi}(m,q)} \le \liminf_{r\to\infty} \frac{\log^{[p]} \nu(r,fog)}{\log^{[m]} \nu(r^A,g)} \le \frac{\lambda^{L}_{fog,\Psi}(p,q)}{\lambda^{L}_{g,\Psi}(m,q)} \le \limsup_{r\to\infty} \frac{\log^{[p]} \nu(r,fog)}{\log^{[m]} \nu(r^A,g)} \le \frac{\rho^{L}_{fog,\Psi}(p,q)}{\lambda^{L}_{g,\Psi}(m,q)}.$$

$$(iii)\liminf_{r\to\infty}\frac{\log^{[p]}\nu(r,fog)}{\log^{[m]}\nu(r^A,g)}\le\min\left\{\frac{\lambda^L_{fog,\Psi}(p,q)}{\lambda^L_{g,\Psi}(m,q)},\frac{\rho^L_{fog,\Psi}(p,q)}{\rho^L_{g,\Psi}(m,q)}\right\}$$

$$\le\max\left\{\frac{\lambda^L_{fog,\Psi}(p,q)}{\lambda^L_{g,\Psi}(m,q)},\frac{\rho^L_{fog,\Psi}(p,q)}{\rho^L_{g,\Psi}(m,q)}\right\}\le\limsup_{r\to\infty}\frac{\log^{[p]}\nu(r,fog)}{\log^{[m]}\nu(r^A,g)}.$$

Proof : (i)From the definition of L-(p,q)th order we have for arbitrary positive ε and for all large values of r,

$$\log^{[p]}\nu(r,fog)\le\left(\rho^L_{fog,\Psi}(p,q)+\varepsilon\right)\log^{[q]}[\Psi(r)L(r)]\text{.................................}(1)$$

and for a sequence of values of r tending to infinity,

$$\log^{[m]}\nu(r\ \ ,g)\ge(\rho^L_{g,\Psi}(m,q)-\varepsilon)\log^{[q]}[\Psi(r)L(r)]$$

$$\Rightarrow\log^{[m]}\nu(r^A,g)\ge(\rho^L_{g,\Psi}(m,q)-\varepsilon)\log^{[q]}[\Psi(r)L(r)]+O(1)\text{................................}(2)$$

Now from (1) and (2) it follows for a sequence of values of r tending to infinity,

$$\frac{\log^{[p]}\nu(r,fog)}{\log^{[m]}\nu(r^A,g)}\le\frac{\rho^L_{fog,\Psi}(p,q)+\varepsilon}{\rho^L_{g,\Psi}(m,q)-\varepsilon}+O(1).$$

As $\varepsilon>0$ is arbitrary we obtain that

$$\liminf_{r\to\infty}\frac{\log^{[p]}\nu(r,fog)}{\log^{[m]}\nu(r^A,g)}\le\frac{\rho^L_{fog,\Psi}(p,q)}{\rho^L_{g,\Psi}(m,q)}\text{.............................}(3)$$

Now for a sequence of values of r tending to infinity,

$$\log^{[p]}\nu(r,fog)\ge\left(\rho^L_{fog,\Psi}(p,q)-\varepsilon\right)\log^{[q]}[\Psi(r)L(r)]\text{...................}(4)$$

Also for sufficiently large values of r,

$$\log^{[m]}\nu(r\ \ ,g)\le\left(\rho^L_{g,\Psi}(m,q)+\varepsilon\right)\log^{[q]}[\Psi(r)L(r)]$$

$$\Rightarrow\log^{[m]}\nu(r^A,g)\le\left(\rho^L_{g,\Psi}(m,q)+\varepsilon\right)\log^{[q]}[\Psi(r)L(r)]+O(1)\text{................}(5)$$

So combining (4) and (5) we get for a sequence of values of r tending to infinity,

$$\frac{\log^{[p]}\nu(r,fog)}{\log^{[m]}\nu(r^A,g)}\ge\frac{\rho^L_{fog,\Psi}(p,q)-\varepsilon}{\rho^L_{g,\Psi}(m,q)+\varepsilon}+O(1).$$

As $\varepsilon>0$ is arbitrary it follows that

$$\limsup_{r\to\infty}\frac{\log^{[p]}\nu(r,fog)}{\log^{[m]}\nu(r^A,g)}\ge\frac{\rho^L_{fog,\Psi}(p,q)}{\rho^L_{g,\Psi}(m,q)}\text{.......................................}(6)$$

Thus (i) follows from (3) and (6).

(ii)From the definition of L -(p,q)[th] lower order we have for arbitrary $\varepsilon > 0$ and for all large values of r,

$$\log^{p} \nu(r\ , fog) \geq (\lambda^{L}_{fog,\Psi}(p,q)-\varepsilon)\log^{[q]}[\Psi(r)L(r)] \quad \text{..............................(7)}$$

Now from (7) and (5) it follows for all large values of r,

$$\frac{\log^{[p]}\nu(r, fog)}{\log^{[m]}\nu(r^{A}, g)} \geq \frac{\lambda^{L}_{fog,\Psi}(p,q)-\varepsilon}{\rho^{L}_{g,\Psi}(m,q)+\varepsilon}+O(1).$$

As $\varepsilon > 0$ is arbitrary we obtain that

$$\liminf_{r\to\infty} \frac{\log^{[p]}\nu(r, fog)}{\log^{[m]}\nu(r^{A}, g)} \geq \frac{\lambda^{L}_{fog,\Psi}(p,q)}{\rho^{L}_{g,\Psi}(m,q)} \quad \text{..........................(8)}$$

Again for a sequence of values of r tending to infinity,

$$\log^{[p]} \nu(r\ , fog) \leq (\lambda^{L}_{fog,\Psi}(p,q)+\varepsilon)\log^{[q]}[\Psi(r)L(r)] \quad \text{..............................(9)}$$

and for all large values of r,

$$\log^{[m]} \nu(r^{A}, g) \geq (\lambda^{L}_{g,\Psi}(m,q)-\varepsilon)\log^{[q]}[\Psi(r)L(r)]+O(1) \quad \text{..............................(10)}$$

So from (9) and (10) we get for a sequence of values of r tending to infinity,

$$\frac{\log^{[p]}\nu(r, fog)}{\log^{[m]}\nu(r^{A}, g)} \leq \frac{\lambda^{L}_{fog,\Psi}(p,q)+\varepsilon}{\lambda^{L}_{g,\Psi}(m,q)-\varepsilon}.$$

As $\varepsilon > 0$ is arbitrary we obtain that

$$\liminf_{r\to\infty} \frac{\log^{[p]}\nu(r, fog)}{\log^{[m]}\nu(r^{A}, g)} \leq \frac{\lambda^{L}_{fog,\Psi}(p,q)}{\lambda^{L}_{g,\Psi}(m,q)} \quad \text{.......................(.11)}$$

Again for a sequence of values of r tending to infinity,

$$\log^{[m]} \nu(r^{A}, g) \leq (\lambda^{L}_{g,\Psi}(m,q)+\varepsilon)\log^{[q]}[\Psi(r)L(r)]+O(1) \quad \text{..............................(12)}$$

Now from (7) and (12) we obtain for a sequence of values of r tending to infinity,

$$\frac{\log^{[p]}\nu(r, fog)}{\log^{[m]}\nu(r^{A}, g)} \geq \frac{\lambda^{L}_{fog,\Psi}(p,q)-\varepsilon}{\lambda^{L}_{g,\Psi}(m,q)+\varepsilon}+O(1).$$

As $\varepsilon > 0$ is arbitrary we obtain that

$$\limsup_{r\to\infty} \frac{\log^{[p]}\nu(r, fog)}{\log^{[m]}\nu(r^{A}, g)} \geq \frac{\lambda^{L}_{fog,\Psi}(p,q)}{\lambda^{L}_{g,\Psi}(m,q)} \quad \text{..........................13)}$$

Again from (1) and (10) it follows for all large values of r,

$$\frac{\log^{[p]}\nu(r, fog)}{\log^{[m]}\nu(r^{A}, g)} \leq \frac{\rho^{L}_{fog,\Psi}(p,q)+\varepsilon}{\lambda^{L}_{g,\Psi}(m,q)-\varepsilon}+O(1).$$

As $\varepsilon > 0$ is arbitrary we obtain that

$$\limsup_{r\to\infty} \frac{\log^{[p]}\nu(r, fog)}{\log^{[m]}\nu(r^{A}, g)} \leq \frac{\rho^{L}_{fog,\Psi}(p,q)}{\lambda^{L}_{g,\Psi}(m,q)} \quad \text{..............................(14)}$$

Thus (ii) follows from (8) , (11) , (13) and (14).

(iii) Combining (i) and (ii) of the theorem, (iii) follows.

Remark 1 : The middle part of the inequality (i) is independent of the constant A.

Example 1: Let $f=\log z, g=e^{e^z}, p=2, q=1, m=3, \Psi=z^2, L=p\exp\left(\frac{1}{r}\right)$ and A= 1 , is a positive real number.

Then

$$\rho^{L}_{fog,\Psi}(p,q)=\frac{1}{2} \text{ and } \lambda^{L}_{g,\Psi}(m,q)=\frac{1}{2},$$

$$\Rightarrow \frac{\rho^{L}_{fog,\Psi}(p,q)}{\lambda^{L}_{g,\Psi}(m,q)}=\frac{1/2}{1/2}=1.$$

Then $\liminf\limits_{r\to\infty}\frac{\log^{[p]}\nu(r,fog)}{\log^{[m]}\nu(r^A,g)}=\liminf\limits_{r\to\infty}\frac{\log^{[2]}\nu(r,e^z)}{\log^{[3]}\nu(r\ ,e^{e^z})}=\liminf\limits_{r\to\infty}\frac{\log r}{\log r}=1,$

$$\limsup_{r\to\infty}\frac{\log^{[p]}\nu(r,fog)}{\log^{[m]}\nu(r^A,g)}=\limsup_{r\to\infty}\frac{\log^{[2]}\nu(r,e^z)}{\log^{[3]}\nu(r\ ,e^{e^z})}=\limsup_{r\to\infty}\frac{\log r}{\log r}=1.$$

So $(i)\liminf\limits_{r\to\infty}\frac{\log^{[p]}\nu(r,fog)}{\log^{[m]}\nu(r^A,g)}=1=\frac{\rho^{L}_{fog,\Psi}(p,q)}{\lambda^{L}_{g,\Psi}(m,q)}=1=\limsup\limits_{r\to\infty}\frac{\log^{[p]}\nu(r,fog)}{\log^{[m]}\nu(r^A,g)}.$

$$(ii)\frac{\lambda^{L}_{fog,\Psi}(p,q)}{\rho^{L}_{g,\Psi}(m,q)}=\frac{1/2}{1/2}=1=\liminf_{r\to\infty}\frac{\log^{[p]}\nu(r,fog)}{\log^{[m]}\nu(r^A,g)}=\frac{\lambda^{L}_{fog,\Psi}(p,q)}{\lambda^{L}_{g,\Psi}(m,q)}=1$$

$$=\limsup_{r\to\infty}\frac{\log^{[p]}\nu(r,fog)}{\log^{[m]}\nu(r^A,g)}=\frac{\rho^{L}_{fog,\Psi}(p,q)}{\lambda^{L}_{g,\Psi}(m,q)}=1.$$

Theorem 2 : If f and g two entire functions with $\rho^{L}_{g,\Psi}(m,q)<\infty$ and $\rho^{L}_{fog,\Psi}(p,q)=\infty$, then for every positive number A ,

$$\limsup_{r\to\infty}\frac{\log^{[p]}\nu(r,fog)}{\log^{[m]}\nu(r^A,g)}=\infty,$$

where p ,q ,m are positive integers with q< min {p , m}.

Proof : Let us assume that the conclusion of the theorem does not hold. Then there exists a constant C>0 such that for all sufficiently large values of r,

$$\log^{[p]}\nu(r,fog)\le C.\log^{[m]}\nu(r^A,g)\qquad\qquad(15)$$

Again from the definition of $\rho^{L}_{g,\Psi}(m,q)$, it follows that

$$\log^{[m]}\nu(r\ ,g)\le(\rho^{L}_{g,\Psi}(m,q)+\varepsilon)\log^{[q]}[\Psi(r)L(r)]$$

$$\Rightarrow \log^{[m]}\nu(r^A,g)\le(\rho^{L}_{g,\Psi}(m,q)+\varepsilon)\log^{[q]}[\Psi(r)L(r)]+O(1)\qquad\qquad(16)$$

holds for all large values of r. So from (15) and (16) we obtain for all sufficiently large values of r,

$$\log^{[p]} \nu(r, fog) \le (\rho^{L}_{g,\Psi}(m,q)+\varepsilon).C.\log^{[q]}[\Psi(r)L(r)]+O(1) \quad \text{.............................(17)}$$

$$\Rightarrow \frac{\log^{[p]} \nu(r, fog)}{\log^{[q]}[\Psi(r)L(r)]} \le (\rho^{L}_{g,\Psi}(m,q)+\varepsilon).C \quad \text{...(18)}$$

From (18) it follows that $\rho^{L}_{fog,\Psi}(p,q) < \infty$.

So we arrive at a contradiction. This proves the theorem.

Example 2: Let $f = z, g = e^{e^{z}}, p = 2, q = 1, m = 3, \Psi = z^{2}, L = p\exp\left(\frac{1}{r}\right)$ and A= 1 , is a positive real number.

Then

$$\rho^{L}_{g,\Psi}(m,q) = \frac{\log^{[m]} \nu(r,g)}{\log^{[q]}[\Psi(r)L(r)]}$$

$$\le \frac{\log^{[3]} M(r,g)}{\log \ [r^{2} L(r)]}$$

$$= \frac{\log^{[3]} e^{e^{z}}}{2\log r + \log p \exp \frac{1}{r}}$$

$$= \frac{\log r}{2\log r + \frac{1}{r} + \log p} = \frac{1}{2},$$

Again, $\rho^{L}_{g,\Psi}(m,q) = \dfrac{\log^{[m]} \nu(R,g)}{\log^{[q]}[\Psi(R)L(R)]}$

$$\ge \frac{\log^{[3]} \frac{R-r}{R} M(r,g)}{\log \ [R^{2} L(R)]}$$

$$= \frac{\log^{[3]} e^{e^{z}} + O(1)}{2\log r + \frac{1}{r} + O(1)}$$

$$= \frac{1}{2}.$$

i.e. $\rho^{L}_{g,\Psi}(m,q) = \frac{1}{2}$.

In a similar way, $\rho^{L}_{fog,\Psi}(p,q) = \infty$.

Now $$\limsup_{r\to\infty} \frac{\log^{[p]} \nu(r, fog)}{\log^{[m]} \nu(r^{A}, g)} = \limsup_{r\to\infty} \frac{\log^{[2]} \nu(r, e^{e^{z}})}{\log^{[3]} \nu(r, e^{e^{z}})} = \limsup_{r\to\infty} \frac{r}{\log r} = \infty.$$

Remark 1 : The second condition is necessary. As we see in the following example.

Example 3 : Let $f = \log z, g = e^{e^{z}}, p = 2, q = 1, m = 3, \Psi = z^{2}, L = p\exp\left(\frac{1}{r}\right)$ and A= 1 , is a positive real number.

Then $\rho^{L}_{g,\Psi}(m,q) = \frac{1}{2} < \infty,$

$\rho^{L}_{fog,\Psi}(p,q) = \frac{1}{2} \ne \infty,$

Now,

$$\limsup_{r\to\infty} \frac{\log^{[p]} \nu(r, fog)}{\log^{[m]} \nu(r^{A}, g)} = \limsup_{r\to\infty} \frac{\log r}{\log r} = 1 \ne \infty.$$

Remark 2 : If we take $\rho^{L}_{f,\Psi}(m,q)<\infty$ instead of $\rho^{L}_{g,\Psi}(m,q)<\infty$ in the above theorem and other conditions remain the same then the theorem remains valid with g replaced by f in the denominator as we see in the following theorem.

Theorem 3 : If f , g be two entire functions with $\rho^{L}_{f,\Psi}(m,q)<\infty$ and $\rho^{L}_{fog,\Psi}(p,q)=\infty$,then for every positive number A,

$$\limsup_{r\to\infty}\frac{\log^{[p]}\nu(r,fog)}{\log^{[m]}\nu(r^{A},f)}=\infty,$$

Where p, q, m are positive integers with q <min{p, m}.

Proof : Let us assume that the conclusion of the theorem does not hold. Then there exists a constant C such that for all sufficiently large values of r ,

$$\log^{[p]}\nu(r,fog)\le C\log^{[m]}\nu(r^{A},f)\quad\ldots\ldots(20)$$

Again from the definition of $\rho^{L}_{f,\Psi}(m,q)$ it follows that

$$\log^{[m]}\nu(r^{A},f)\le(\rho^{L}_{g,\Psi}(m,q)+\varepsilon)\log^{[q]}[\Psi(r)L(r)]+O(1)\quad\ldots\ldots(21)$$

holds for all large values of r. So from (20) and (21) we obtain for all sufficiently large values of r,

$$\log^{[p]}\nu(r,fog)\le(\rho^{L}_{g,\Psi}(m,q)+\varepsilon).C.\log^{[q]}[\Psi(r)L(r)]+O(1)\quad\ldots\ldots(22)$$

From (22) it follows that $\rho^{L}_{fog,\Psi}(p,q)<\infty.$

Thus we arrive at a contradiction. So the theorem is established.

In the line of Theorem 1 and 2 we may respectively state the following two theorems whose proofs are given below.

Theorem 4 : Let f and g be two entire functions such that $0<\lambda^{L^*}_{fog,\Psi}(p,q)\le\rho^{L^*}_{fog,\Psi}(p,q)<\infty$ and $0<\rho^{L^*}_{g,\Psi}(m,q)<\infty,$ where p , q, m are positive integers such that q < min{p,m}. Then for any positive integer A,

$$(i)\liminf_{r\to\infty}\frac{\log^{[p]}\nu(r,fog)}{\log^{[m]}\nu(r^{A},g)}\le\frac{\rho^{L^*}_{fog,\Psi}(p,q)}{\rho^{L^*}_{g,\Psi}(m,q)}\le\limsup_{r\to\infty}\frac{\log^{[p]}\nu(r,fog)}{\log^{[m]}\nu(r^{A},g)}.$$

Further if $\lambda^{L^*}_{g,\Psi}(m,q)>0$ then

$$(ii)\ \frac{\lambda^{L^*}_{fog,\Psi}(p,q)}{\rho^{L^*}_{g,\Psi}(m,q)}\le\liminf_{r\to\infty}\frac{\log^{[p]}\nu(r,fog)}{\log^{[m]}\nu(r^{A},g)}\le\frac{\lambda^{L^*}_{fog,\Psi}(p,q)}{\lambda^{L^*}_{g,\Psi}(m,q)}\le\limsup_{r\to\infty}\frac{\log^{[p]}\nu(r,fog)}{\log^{[m]}\nu(r^{A},g)}\le\frac{\rho^{L^*}_{fog,\Psi}(p,q)}{\lambda^{L^*}_{g,\Psi}(m,q)}.$$

$$(iii)\liminf_{r\to\infty}\frac{\log^{[p]}\nu(r,fog)}{\log^{[m]}\nu(r^{A},g)}\le\min\left\{\frac{\lambda^{L^*}_{fog,\Psi}(p,q)}{\lambda^{L^*}_{g,\Psi}(m,q)},\frac{\rho^{L^*}_{fog,\Psi}(p,q)}{\rho^{L^*}_{g,\Psi}(m,q)}\right\}$$

$$\le\max\left\{\frac{\lambda^{L^*}_{fog,\Psi}(p,q)}{\lambda^{L^*}_{g,\Psi}(m,q)},\frac{\rho^{L^*}_{fog,\Psi}(p,q)}{\rho^{L^*}_{g,\Psi}(m,q)}\right\}\le\limsup_{r\to\infty}\frac{\log^{[p]}\nu(r,fog)}{\log^{[m]}\nu(r^{A},g)}.$$

Proof : (i)From the definition of L*-(p,q)th order we have for arbitrary positive ε and for all large values of r,

$$\log^{[p]}\nu(r, fog)\leq\left(\rho^{L^*}_{fog,\Psi}(p,q)+\varepsilon\right)\log^{[q]}\left[\Psi(r)e^{L(r)}\right]\ldots\ldots(23)$$

and for a sequence of values of r tending to infinity,

$$\log^{[m]}\nu(r^A,g)\geq\left(\rho^{L^*}_{g,\Psi}(m,q)-\varepsilon\right)\log^{[q]}\left[\Psi(r)e^{L(r)}\right]+O(1)\ldots\ldots(24)$$

Now from (23) and (24) it follows for a sequence of values of r tending to infinity,

$$\frac{\log^{[p]}\nu(r, fog)}{\log^{[m]}\nu(r^A,g)}\leq\frac{\rho^{L^*}_{fog,\Psi}(p,q)+\varepsilon}{\rho^{L^*}_{g,\Psi}(m,q)-\varepsilon}+O(1).$$

As $\varepsilon>0$ is arbitrary we obtain that

$$\liminf_{r\to\infty}\frac{\log^{[p]}\nu(r, fog)}{\log^{[m]}\nu(r^A,g)}\leq\frac{\rho^{L^*}_{fog,\Psi}(p,q)}{\rho^{L^*}_{g,\Psi}(m,q)}\ldots\ldots(25)$$

Now for a sequence of values of r tending to infinity,

$$\log^{[p]}\nu(r, fog)\geq\left(\rho^{L^*}_{fog,\Psi}(p,q)-\varepsilon\right)\log^{[q]}\left[\Psi(r)e^{L(r)}\right]\ldots\ldots(26)$$

Also for sufficiently large values of r,

$$\log^{[m]}\nu(r^A,g)\leq\left(\rho^{L^*}_{g,\Psi}(m,q)+\varepsilon\right)\log^{[q]}\left[\Psi(r)e^{L(r)}\right]+O(1)\ldots\ldots(27)$$

So combining (26) and (27) we get for a sequence of values of r tending to infinity,

$$\frac{\log^{[p]}\nu(r, fog)}{\log^{[m]}\nu(r^A,g)}\geq\frac{\rho^{L^*}_{fog,\Psi}(p,q)-\varepsilon}{\rho^{L^*}_{g,\Psi}(m,q)+\varepsilon}+O(1).$$

As $\varepsilon>0$ is arbitrary it follows that

$$\limsup_{r\to\infty}\frac{\log^{[p]}\nu(r, fog)}{\log^{[m]}\nu(r^A,g)}\geq\frac{\rho^{L^*}_{fog,\Psi}(p,q)}{\rho^{L^*}_{g,\Psi}(m,q)}\ldots\ldots(28)$$

Thus (i) follows from (25) and (28).

(ii)From the definition of L*-(p,q)th lower order we have for arbitrary $\varepsilon>0$ and for all large values of r,

$$\log^{[p]}\nu(r, fog)\geq\left(\lambda^{L^*}_{fog,\Psi}(p,q)-\varepsilon\right)\log^{[q]}\left[\Psi(r)e^{L(r)}\right]+O(1)\ldots\ldots(29)$$

Now from (27) and (29) it follows for all large values of r,

$$\frac{\log^{[p]}\nu(r, fog)}{\log^{[m]}\nu(r^A,g)}\geq\frac{\lambda^{L^*}_{fog,\Psi}(p,q)-\varepsilon}{\rho^{L^*}_{g,\Psi}(m,q)+\varepsilon}+O(1).$$

As $\varepsilon>0$ is arbitrary we obtain that

$$\liminf_{r\to\infty}\frac{\log^{[p]}\nu(r, fog)}{\log^{[m]}\nu(r^A,g)}\geq\frac{\lambda^{L^*}_{fog,\Psi}(p,q)}{\rho^{L^*}_{g,\Psi}(m,q)}\ldots\ldots(30)$$

Again for a sequence of values of r tending to infinity,

$$\log^{[p]}\nu(r, fog)\leq\left(\lambda^{L^*}_{fog,\Psi}(p,q)+\varepsilon\right)\log^{[q]}\left[\Psi(r)e^{L(r)}\right]+O(1)\ldots\ldots(31)$$

and for all large values of r,

$$\log^{[m]} \nu(r^A, g) \geq (\lambda^{L^*}_{g,\Psi}(m,q) - \varepsilon) \log^{[q]}\left[\Psi(r)e^{L(r)}\right] + O(1) \quad \text{.................................(32)}$$

So from (31) and (32) we get for a sequence of values of r tending to infinity,

$$\frac{\log^{[p]} \nu(r, fog)}{\log^{[m]} \nu(r^A, g)} \leq \frac{\lambda^{L^*}_{fog,\Psi}(p,q) + \varepsilon}{\lambda^{L^*}_{g,\Psi}(m,q) - \varepsilon} + O(1).$$

As $\varepsilon > 0$ is arbitrary we obtain that

$$\liminf_{r\to\infty} \frac{\log^{[p]} \nu(r, fog)}{\log^{[m]} \nu(r^A, g)} \leq \frac{\lambda^{L^*}_{fog,\Psi}(p,q)}{\lambda^{L^*}_{g,\Psi}(m,q)} \quad \text{...........................(33)}$$

Again for a sequence of values of r tending to infinity,

$$\log^{[m]} \nu(r^A, g) \leq (\lambda^{L^*}_{g,\Psi}(m,q) + \varepsilon) \log^{[q]}\left[\Psi(r)e^{L(r)}\right] + O(1) \quad \text{.................................(34)}$$

Now from (29) and (34) we obtain for a sequence of values of r tending to infinity,

$$\frac{\log^{[p]} \nu(r, fog)}{\log^{[m]} \nu(r^A, g)} \geq \frac{\lambda^{L^*}_{fog,\Psi}(p,q) - \varepsilon}{\lambda^{L^*}_{g,\Psi}(m,q) + \varepsilon} + O(1).$$

As $\varepsilon > 0$ is arbitrary we obtain that

$$\limsup_{r\to\infty} \frac{\log^{[p]} \nu(r, fog)}{\log^{[m]} \nu(r^A, g)} \geq \frac{\lambda^{L^*}_{fog,\Psi}(p,q)}{\lambda^{L^*}_{g,\Psi}(m,q)} \quad \text{...........................(35)}$$

Again from (23) and (32) it follows for all large values of r

$$\frac{\log^{[p]} \nu(r, fog)}{\log^{[m]} \nu(r^A, g)} \leq \frac{\rho^{L^*}_{fog,\Psi}(p,q) + \varepsilon}{\lambda^{L^*}_{g,\Psi}(m,q) - \varepsilon} + O(1).$$

As $\varepsilon > 0$ is arbitrary we obtain that

$$\limsup_{r\to\infty} \frac{\log^{[p]} \nu(r, fog)}{\log^{[m]} \nu(r^A, g)} \leq \frac{\rho^{L^*}_{fog,\Psi}(p,q)}{\lambda^{L^*}_{g,\Psi}(m,q)} \quad \text{.................................(36)}$$

Thus (ii) follows from (30) , (33) , (35) and (36).

(iii) Combining (i) and (ii) of the theorem, (iii) follows.

Theorem 5 If f and g two entire functions with $\rho^{L^*}_{g,\Psi}(m,q) < \infty$ and $\rho^{L^*}_{fog,\Psi}(p,q) = \infty$, then for every positive number A ,

$$\limsup_{r\to\infty} \frac{\log^{[p]} \nu(r, fog)}{\log^{[m]} \nu(r^A, g)} = \infty,$$

where p ,q ,m are positive integers with q< min {p , m}.

Proof : Let us assume that the conclusion of the theorem does not hold. Then there exists a constant C>0 such that for all sufficiently large values of r,

$$\log^{[p]} \nu(r, fog) \leq C.\log^{[m]} \nu(r^A, g) \quad \text{...(37)}$$

Again from the definition of $\rho^{L^*}_{g,\Psi}(m,q)$, it follows that

$$\log^{[m]} \nu(r^A, g) \leq (\rho^{L^*}_{g,\Psi}(m,q)+\varepsilon)\log^{[q]}\left[\Psi(r)e^{L(r)}\right]+O(1)\ldots\ldots(38)$$

holds for all large values of r. So from (37) and (38) we obtain for all sufficiently large values of r,

$$\log^{[p]} \nu(r\ , fog) \leq (\rho^{L^*}_{g,\Psi}(m,q)+\varepsilon).C.\log^{[q]}\left[\Psi(r)e^{L(r)}\right]+O(1)\ldots\ldots(39)$$

From (39) it follows that $\rho^{L^*}_{fog,\Psi}(p,q) < \infty.$

So we arrive at a contradiction. This proves the theorem.

Remark 4 : If we take $\rho^{L^*}_{f,\Psi}(m,q) < \infty$ instead of $\rho^{L^*}_{g,\Psi}(m,q) < \infty$ in this theorem and the other conditions remain same then the theorem remains valid with g replaced byf in the denominator as we see in the following theorem.

Theorem 6 : If f , g be two entire functions with $\rho^{L^*}_{f,\Psi}(m,q) < \infty$ and $\rho^{L^*}_{fog,\Psi}(p,q) = \infty$,then for every positive number A,

$$\limsup_{r\to\infty} \frac{\log^{[p]} \nu(r, fog)}{\log^{[m]} \nu(r^A, f)} = \infty,$$

Where p, q, m are positive integers with q <min{p, m}.

Proof : Let us assume that the conclusion of the theorem does not hold. Then there exists a constant C such that for all sufficiently large values of r ,

$$\log^{[p]} \nu(r, fog) \leq C \log^{[m]} \nu(r^A, f)\ldots\ldots(40)$$

Again from the definition of $\rho^{L^*}_{f,\Psi}(m,q)$ it follows that

$$\log^{[m]} \nu(r^A, f) \leq (\rho^{L^*}_{g,\Psi}(m,q)+\varepsilon)\log^{[q]}\left[\Psi(r)e^{L(r)}\right]+O(1)\ldots\ldots(41)$$

holds for all large values of r. So from (40) and (41) we obtain for all sufficiently large values of r,

$$\log^{[p]} \nu(r, fog) \leq (\rho^{L^*}_{g,\Psi}(m,q)+\varepsilon).C.\log^{[q]}\left[\Psi(r)e^{L(r)}\right]+O(1)\ldots\ldots(42)$$

From (42) it follows that $\rho^{L^*}_{fog,\Psi}(p,q) < \infty.$

Thus we arrive at a contradiction. So the theorem is established.

References

[1] Filevych, P.V. , "On the growth of maximum modulous of an entire function depending on the growth of its central index" , *Ufa Mathematical Journal*, Vol. 3 , No. 1, pp. 92-100,2011.

[2] Juneja , O.P.,KapoorG. P.andBajpai S.K. , " On the (p,q)- order and lower (p,q)-order of an entire function" ,*J. ReineAngew. Math.*,Vol. 282 , pp. 99-102,1976.

[3] Singh,A.P. , " Growth of composite entire functions" , *Kodai Math. J.*, Vol. 8 , pp. 99-102,1985.

[4] Somasundaram,D. and Thamizharasi R., "A note on the entire functions of L-bounded index and L-type" , *Indian J. Pure. Appl. Math.* , Vol. 19 , No. 3 , pp. 284-293,March 1988.